<u>Exercise</u> 1. Let A be a unital C*-algebra, H_1, H_2 Hilbert spaces, $\phi_1 : A \to BL(H_1, H_1)$ and $\phi_2 : A \to BL(H_2, H_2)$ cyclic representations. Suppose that $\langle \phi_1(a)\psi_1, \psi_1\rangle_1 = \langle \phi_2(a)\psi_2, \psi_2\rangle_2$ for all $a \in A$, where ψ_1, ψ_2 are the cyclic vectors in H_1 and H_2 respectively. Show that there exists a unitary operator (i.e., an invertible linear isometry) $W : H \to K$ such that $\phi(a) = W^*\psi(a)W$ for all $a \in A$.

5 Spectral decomposition of normal operators

Proposition 5.1 (Spectral Theorem for Normal Elements)**.** *Let A be a unital C*-algebra and $a \in A$ normal. Then, there exists an isometric embedding of unital *-algebras $\phi : C(\sigma_A(a), \mathbb{C}) \to A$ such that $\phi(\mathrm{id}) = a$.*

Proof. **<u>Exercise</u>.** ☐

Of course, an important application of this is the case when A is the algebra of bounded operators on some Hilbert space and a is a normal operator.

In the context of this proposition we also use the notation $f(a) := \phi(f)$ for $f \in C(\sigma_A(a), \mathbb{C})$. We use the same notation if f is defined on a larger subset of the complex plane.

Corollary 5.2 (Continuous Spectral Mapping Theorem)**.** *Let A be a unital C*-algebra, $a \in A$ normal and $f : T \to \mathbb{C}$ continuous such that $\sigma_A(a) \subseteq T$. Then, $\sigma_A(f(a)) = f(\sigma_A(a))$.*

Proof. **<u>Exercise</u>.** ☐

Corollary 5.3. *Let A be a unital C*-algebra and $a \in A$ normal. Furthermore, let $f : \sigma_A(a) \to \mathbb{C}$ and $g : f(\sigma_A(a)) \to \mathbb{C}$ continuous. Then $(g \circ f)(a) = g(f(a))$.*

Proof. **<u>Exercise</u>.** ☐

Definition 5.4. Let A be a unital C*-algebra. If $u \in A$ is invertible and satisifes $u^* = u^{-1}$ we call u *unitary.* If $p \in A$ is self-adjoint and satisfies $p^2 = p$ we call it a *projector.* If $a \in A$ is positive and invertible we call it *positive definite.*

<u>Exercise</u>. Let A be a unital C*-algebra.

1. Let $u \in A$ be unitary. What can you say about $\sigma_A(u)$?
2. Let $p \in A$ be a projector. Show that $\sigma_A(p) \subseteq \{0, 1\}$.
3. Let $a \in A$ be normal and $\sigma_A(a) \subset \mathbb{R}$. Show that a is self-adjoint.
4. Let $a \in A$ be invertible. Show that there is a unitary element $u \in A$ and a positive definite element $n \in A$ such that $a = un$.

Proposition 5.5. *Let A be a unital C^*-algebra and $a \in A$ normal. Suppose the spectrum of a is the disjoint union of two non-empty subsets $\sigma_A(a) = s_1 \cup s_2$. Then, there exist $a_1, a_2 \in A$ normal, such that $\sigma_A(a_1) = s_1$ and $\sigma_A(a_2) = s_2$ and $a = a_1 + a_2$. Moreover, $a_1 a_2 = a_2 a_1 = 0$ and a commutes both with a_1 and a_2.*

Proof. **Exercise.** □

Proposition 5.6. *Let H be a Hilbert space and $k \in A := BL(H, H)$ compact. Show that $\sigma_A(k)$ is discrete and bounded and has at most one accumulation point, namely 0. If 0 is accumulation point of $\sigma_A(k)$ then $0 \in \sigma_A(k)$.*

Proof. **Exercise.** Use results of Part I of the course to prove this. □

Proposition 5.7. *Let A be a unital C^*-algebra and $k \in A$ normal. Assume moreover that $\sigma_A(k)$ is discrete and has at most one accumulation point, namely 0. Then, there exists a projector $p_\lambda \in A$ for each $\lambda \in \sigma_A(k)$ such that $p_\lambda p_{\lambda'} = 0$ if $\lambda \neq \lambda'$ and*

$$k = \sum_{\lambda \in \sigma_A(k)} \lambda p_\lambda \quad \text{and} \quad e = \sum_{\lambda \in \sigma_A(k)} p_\lambda.$$

Proof. **Exercise.** (Explain also in which sense the sums converge!) □